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Interpolation and approximation from convex sets. II. Infinite-dimensional interpolation

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Dedicated to Prof. Larry L. Schumaker on the occasion of his 60th birthday

Abstract

Let X and Y be topological vector spaces, A be a continuous linear map from X to Y, $C \subset X$, B be a convex set dense in C, and $d \in Y$ be a data point. Conditions are derived guaranteeing the set $B \cap A^{-1}(d)$ to be nonempty and dense in $C \cap A^{-1}(d)$. The paper generalizes earlier results by the authors to the case where Y is infinite dimensional. The theory is illustrated with two examples concerning the existence of smooth monotone extensions of functions defined on a domain of the Euclidean space to a larger domain.

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1. Introduction

Let X and Y be topological vector spaces, C be a subset of X, and let $A : X \to Y$ be a continuous linear map. We are interested in the following abstract interpolation problem: given $d \in Y$,

$$\text{find an } x \in C \text{ such that } Ax = d. \tag{1}$$

If C is a proper subset of X, one often refers to this problem as a *constrained interpolation problem*. The set C can then be viewed as a set defining the constraints that we wish to impose on the element x, in addition to the requirements that it belongs to an appropriate space X and that it interpolates

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the *data point* $d \in Y$. For example, X could be a function space, such as a space of continuous functions or a Sobolev space, and Ax the restriction of $x \in X$ to a subset of the domain of x. Next to the requirement $Ax = d$, it may be also desired that x has a certain "shape", which can be enforced by defining the set C appropriately. Hence, if C is a set of monotone functions, say, any solution x of the interpolation problem (1) will be a monotone function. Since the constraints imposed on x often involve the consideration of the shape of x, constrained interpolation is sometimes also coined *shape preserving interpolation*.

If problem (1) has a solution for a given data point d, we say that d is *admissible*. Clearly, this means that $d \in A[C] := \{Ax: x \in C\}$, or equivalently

$$C \cap A^{-1}(d) \neq \emptyset,$$

where

$$A^{-1}(d) := \{x \in X: Ax = d\}.$$

In [7] we studied the problem of solvability of constrained interpolation when the set C is replaced by a convex subset B of C. In this case the constrained interpolation problem consists in finding an element x from X such that

$$x \in B \cap A^{-1}(d).$$

Frequently, B is given as the intersection of C with a linear subspace S of X, such as a space of smooth functions. Alternatively, B could represent a set of elements that are strictly contained in C, in some sense. For example, if C is a set of monotone functions, B could be the set of all "strictly" monotone functions in C.

It is clear that even if d admits interpolation from C, it may not admit interpolation from B. However, under the assumptions that Y is finite-dimensional and B dense in C, we proved in [7] that each *interior data point* $d \in \operatorname{int}(A[C])$ admits interpolation from B and that in fact $B \cap A^{-1}(d)$ is dense in $C \cap A^{-1}(d)$. In Section 2, we present new proofs of this result.

The unifying approach introduced in the above-cited paper has many interesting applications in the area of constrained interpolation and approximation. However, the tools established in that paper are, thus far, useful only in the setting of a finite-dimensional space Y. Our aim in this paper is to extend the results of [7] so as to dispense with this restriction on Y. If Y is infinite dimensional, we shall refer to (1) as *infinite-dimensional interpolation* or simply *infinite interpolation*. Infinite interpolation, sometimes also called "transfinite interpolation", arises in various areas of mathematics, including the spline theory and computer-aided geometric design. In particular, *extension problems* can be viewed as instances of infinite interpolation. The objective there is to prove the existence of an extension of a function, defined on a given domain, to a larger domain, such that the extension exhibits certain desired properties. For example, suppose that a function is specified on the boundary of a convex domain in the plane, such as the unit square. It was shown in [3] that, roughly, if the function on the boundary is smooth and consistent with a monotonicity requirement, then the function can be extended as a smooth monotone function on the square.

To illustrate the usefulness of the theoretical results obtained in this paper, we shall make use, among other things, of the open mapping theorem along with the assumption that the set B has nonempty interior in an "intermediate" subspace S of X, which is continuously imbedded in X. While this assumption may seem quite strong, it nevertheless allows us to apply our results successfully to the above-mentioned problem of smooth monotone extension of boundary data, see Section 5.

Lastly, we mention that the proofs of some of our results can be simplified if we assume that X is locally convex. We shall indicate such possible simplifications in the pertinent proofs. As a matter of fact, the requirement of local convexity would not be too prohibitive since it is met in most applications. However, to be consistent with [7], we shall present the results for the general case, which in turn will require some additional considerations not needed for locally convex spaces.

2. Preliminary results

All spaces considered in this paper will be assumed real and separated (Hausdorff). Throughout, we will adopt the notations used in Holmes' book [5]. For convenience, we recall some basic algebraic and topological concepts.

Let L be a topological vector space. The *line segment* joining the elements $x \neq y$ of L is denoted by $[x, y] = \{(1-t)x + ty : 0 \leqslant t \leqslant 1\}$, and $(x, y) = [x, y] \setminus \{x, y\}$. The *core* $\operatorname{cor}(M)$ of $M \subset L$ consists of all elements $x \in M$ such that for each $y \in L$, $y \neq x$, there exists $z \in (x, y)$ for which $[x, z] \subset M$. If $0 \in \operatorname{cor}(M)$, the set M is called *absorbing*. Furthermore, M is *balanced* if $tM \subset M$ for each $|t| \leqslant 1$. For each open neighborhood U of $0 \in L$, there exists a balanced open 0-neighborhood V such that $V + V \subset U$. The *closure* of a set $M \subset L$ is denoted by $\operatorname{cl}(M)$, and its (topological) *interior* by $\operatorname{int}(M)$.

It is well known that all open neighborhoods of the origin in L are absorbing, hence always

$$\operatorname{int}(M) \subset \operatorname{cor}(M).$$

If the space Y is finite-dimensional, the interior and the core agree for convex sets, but this is not true in general for infinite-dimensional Y. However, if a convex set M is *solid*, i.e., $\operatorname{int}(M) \neq \emptyset$, then

$$\operatorname{int}(M) = \operatorname{cor}(M), \quad \operatorname{cl}(M) = \operatorname{cl}(\operatorname{int}(M)), \quad \operatorname{int}(M) = \operatorname{int}(\operatorname{cl}(M)). \tag{2}$$

We say that B is *dense* in C if

$$B \subset C \subset \operatorname{cl}(B).$$

Our main concern in this paper is to generalize the following two results from [7] to the case where the range of the continuous linear map $A : X \to Y$ is infinite-dimensional.

Theorem 1. *Let Y be finite-dimensional and let B be a dense convex subset of $C \subset X$. Then* $\operatorname{int}(A[C]) = \operatorname{int}(A[B])$.

Thus every interior data point $d \in \operatorname{int}(A[C])$ can be interpolated by elements from B, provided that B is convex and dense in C. The result can be improved in the sense that every interpolant $x \in C$ such that $Ax = d \in \operatorname{int}(A[C])$ can be also simultaneously approximated by elements from B.

Theorem 2. *Let Y be finite dimensional, B be a dense convex subset of $C \subset X$, and let $d \in$* $\operatorname{int}(A[C])$. *Then $B \cap A^{-1}(d)$ is dense in $C \cap A^{-1}(d)$.*

This theorem generalizes the well-known *Singer–Yamabe Theorem* [5], obtained by setting $C = X$ and assuming that B is a dense convex subset of X. Our proof of Theorem 2 given in [7] follows closely the proof of the Singer–Yamabe Theorem in [5, p. 49]. There the result is first established for the case where A is a continuous linear functional and then the general case is proved by induction on the dimension of Y. Clearly, that method of proof cannot be used for infinite-dimensional Y.

Below, we show two alternative proofs of Theorem 2. The first proof reveals some similarities with earlier proofs of related results, in particular with Deutsch's proof of the Singer–Yamabe Theorem [4]. The second proof is based on different arguments and it motivates our generalization of Theorem 2 to the infinite-dimensional case.

First proof of Theorem 2. Let $x \in C$ be such that $d = Ax \in \mathrm{int}(A[C])$. Shifting the origins in both spaces X and Y, if necessary, we can assume without loss of generality that $x = 0$, hence also $d = 0$. Since Y is finite-dimensional, we can identify Y with $\mathbb{R}^n$, for some n. In this new setting, our assumption on d means $0 \in \mathrm{int}(A[C])$, and we must prove that $B \cap A^{-1}(0)$ is dense in $C \cap A^{-1}(0)$. If $\mathrm{int}(A[C]) = \{0\}$, i.e., $n = 0$ and A is the trivial map, the result holds trivially. Otherwise, it will be sufficient to prove that for every open neighborhood U of $0 \in X$, we have $B \cap A^{-1}(0) \cap U \neq \emptyset$. Let $P = \{-1, 1\}^n$ denote the set of extreme points of $[-1, 1]^n$, the closure of $I := (-1, 1)^n$ in $\mathbb{R}^n$. Then it is not difficult to show that there exists an $\varepsilon > 0$ and elements $x_p \in X$ for $p \in P$, such that

(i) $2\varepsilon P \subset A[C]$ and $Ax_p = 2\varepsilon p$, $p \in P$, since $0 \in \mathrm{int}(A[C])$,
(ii) $x_p \in U$, $p \in P$, since U is absorbing,
(iii) $x_p \in \mathrm{cl}(C)$, $p \in P$, since $\mathrm{cl}(C) = \mathrm{cl}(B)$ is convex and $0 \in C \subset \mathrm{cl}(C)$.

We know that there exists a balanced open neighborhood $\tilde{V}$ of $0 \in X$ such that $\tilde{V} + \tilde{V} \subset U$. Hence, by induction there is also a balanced open neighborhood V of $0 \in X$ such that $V + \cdots + V \subset U$, where the sum consists of 2^n terms. By continuity of A, for every $p \in P$ the preimage $A^{-1}[2\varepsilon p + \varepsilon I]$ of the open set $2\varepsilon p + \varepsilon I \subset \mathbb{R}^n$ is an open neighborhood of $x_p \in X$. By the denseness of B in $\mathrm{cl}(C) = \mathrm{cl}(B)$, each open neighborhood $(x_p + V) \cap A^{-1}[2\varepsilon + \varepsilon I]$ of $x_p \in \mathrm{cl}(C)$ contains an element $b_p \in B$.

It is easy to see that $0 \in \mathbb{R}^n$ belongs to the convex hull of $\{Ab_p : p \in P\}$, i.e., $0 = \sum_{p \in P} \alpha_p A b_p$, where $\sum_{p \in P} \alpha_p = 1$ and $\alpha_p \geqslant 0$, $p \in P$. Thus, $b := \sum_{p \in P} \alpha_p b_p \in A^{-1}(0)$, $b \in B$, since B is convex, and $b \in V + \cdots + V \subset U$, since V is balanced. □

Remark 3. The proof given above can be simplified if X is a locally convex space. Then the neighborhood V is not needed, since it is sufficient to consider convex neighborhoods U of $0 \in X$. In the last step of the proof, $b \in U$ follows immediately from $b_p \in U$, $p \in P$.

Remark 4. Similar ideas were used in several papers on comonotone interpolation by polynomials, Müntz polynomials, and, more generally, by dense linear systems [9–12]. However, in all these papers only the interpolation part of our result is established. In terms of the proof given above, the authors of these papers used (i) and (iii) to conclude $b \in B$, but they did not observe that (ii) implies $b \in U$.

Remark 5. We could use the extreme points of a simplex in $\mathbb{R}^n$ instead of the cube $[-1, 1]^n$ in the arguments above, as it was done in [4] for $C = X$ and B a dense linear subspace of X, and also in [11] in the context of comonotone interpolation.

To give the second proof and its generalization, we first isolate a simple, purely algebraic result. The following lemma does not require the assumption of continuity of the map A, nor finite dimensionality of Y.

Lemma 6. *Let $M \subset X$ be a convex set such that $0 \in M$ and $0 \in \mathrm{cor}(A[M])$, and let $U \subset X$ be absorbing. Then $0 \in \mathrm{cor}(A[M \cap U])$.*

Proof. The assumption $0 \in \mathrm{cor}(A[M])$ means that for every $y \in Y$, $y \neq 0$, there exists a $\tilde{y} \in (0, y)$ such that $[0, \tilde{y}] \subset A[M]$. In particular, $\tilde{y} = A\tilde{x}$ for some $\tilde{x} \in M$, $\tilde{x} \neq 0$. Since the set U is absorbing in X, we can find $z \in (0, \tilde{x})$ for which $[0, z] \subset U$. This implies $[0, z] \subset M \cap U$, by the convexity of M, i.e., $[0, Az] \subset A[M \cap U]$. Since y was arbitrary in Y we conclude that $A[M \cap U]$ is absorbing, i.e., $0 \in \mathrm{cor}(A[M \cap U])$. □

To better reveal the main idea of the second proof, in the following we will assume that the space X is locally convex. Later, this restriction will be dropped when we prove a generalization of Theorem 2, see Theorem 10.

Second proof of Theorem 2. As in the first proof, we assume $x = 0$ and $0 \in \mathrm{int}(A[C])$. Now it is sufficient to prove that for every *convex* open neighborhood U of $0 \in X$ we have $B \cap A^{-1}(0) \cap U \neq \emptyset$, i.e., $0 \in A[B \cap U]$. Obviously, the set $B \cap U$ is dense in $\mathrm{cl}(B) \cap U$. If we can show that $0 \in \mathrm{int}(A[\mathrm{cl}(B) \cap U])$, the claim will follow from Theorem 1 applied to the convex sets $B \cap U$ and $\mathrm{cl}(B) \cap U$.

Since $0 \in \mathrm{int}(A[C]) \subset \mathrm{int}(A[\mathrm{cl}(B)]) \subset \mathrm{cor}(A[\mathrm{cl}(B)])$ and the neighborhood U is absorbing, we get $0 \in \mathrm{cor}(A[\mathrm{cl}(B) \cap U])$ by Lemma 6. The proof is finished using the fact that in a finite-dimensional space the interior and the core of a convex set are equal. □

Remark 7. In [7], Theorems 1 and 2 have been stated for data points $d \in \mathrm{ri}(A[C])$, where $\mathrm{ri}(A[C])$ is the *relative interior* of $A[C]$. Recall that the relative interior $\mathrm{ri}(M)$ of a set $M \subset Y$ is its interior with respect to $\mathrm{cl}(\mathrm{aff}(M))$, the closure of the affine hull of M, which for finite-dimensional spaces Y is the same as $\mathrm{aff}(M)$. We briefly describe how this slight generalization follows from the above theorems.

Shifting the origins in both spaces X and Y, we can assume that $0 \in C$. Then $\mathrm{aff}(B) \subset \mathrm{aff}(C) = \mathrm{span}(C) \subset \mathrm{span}(\mathrm{cl}(B)) \subset \mathrm{cl}(\mathrm{aff}(B))$. Hence, the space X in the two theorems can be replaced by $\mathrm{cl}(\mathrm{span}(C))$. Correspondingly, the space Y is replaced by

$$A[\mathrm{aff}(B)] \subset A[\mathrm{span}(C)] \subset A[\mathrm{cl}(\mathrm{aff}(B))] \subset \mathrm{cl}(A[\mathrm{cl}(\mathrm{aff}(B))]) = \mathrm{cl}(A[\mathrm{aff}(B)]),$$

where, in fact, equality holds throughout, see also [2]. The interiors of $A[B]$, $A[C]$, and $\mathrm{cl}(A[\mathrm{cl}(B)]) = \mathrm{cl}(A[B])$ in this new setting correspond to the respective relative interiors in the original setup.

3. Infinite interpolation and simultaneous approximation

In this section we will do away with the restriction that Y is finite-dimensional. The finite dimensionality will be replaced by an assumption on the interplay between the convex set $B \subset X$ and the map A, see Definition 9 below. Sufficient conditions for this assumption to be fulfilled will be provided in Section 4.

We begin with a generalization of Theorem 1.

Theorem 8. *Let B be a convex dense subset of C, such that* $\operatorname{int}(A[B]) \neq \emptyset$. *Then* $\operatorname{int}(A[B]) = \operatorname{int}(A[C])$.

Proof. The proof is similar to the finite-dimensional case [7]. By continuity of A, we have

$$\operatorname{cl}(A[B]) = \operatorname{cl}(A[\operatorname{cl}(B)]).$$

Noting that $A[B]$ is convex, since B is convex, and that it is solid by assumption, we obtain

$$\operatorname{int}(A[B]) = \operatorname{int}(\operatorname{cl}(A[B])) = \operatorname{int}(\operatorname{cl}(A[\operatorname{cl}(B)])) = \operatorname{int}(A[\operatorname{cl}(B)])$$

as a consequence of (2). Since $B \subset C \subset \operatorname{cl}(B)$, it follows that $\operatorname{int}(A[B]) = \operatorname{int}(A[C])$. □

Theorem 8 can be interpreted as saying that for each interior data point $d \in \operatorname{int}(A[C])$ there exists an element $x \in B$ such that $Ax = d$, provided that $A[B]$ is solid. This latter condition may seem quite restrictive since its verification might require a characterization of $\operatorname{int}(A[B])$. However, if we knew such a characterization, Theorem 8 would most likely ascertain nothing more than what might already be a consequence of that characterization. On the other hand, it is obvious that in general an assumption like $\operatorname{int}(A[B]) \neq \emptyset$ cannot be entirely removed. To see this, take for example $C = X = Y$, $B \neq X$ a dense linear subspace of X, and A the identity map. Later we shall argue that in certain cases one can verify the above condition without the need to characterize $\operatorname{int}(A[B])$.

As with Theorem 1, Theorem 8 can also be strengthened so as to simultaneously approximate elements of C, corresponding to interior data points, by elements from B. It will be convenient to make the following definition.

Definition 9. Let A be a continuous linear map from X to Y and let $M \subset X$ be nonempty. The map A is called *open relative to M*, or *M-open* for short, if $\operatorname{int}(A[M \cap U]) \neq \emptyset$ for every open $U \subset X$ such that $M \cap U \neq \emptyset$.

The notion of openness of A relative to a convex set allows us to formulate the following theorem.

Theorem 10. *Let B be a dense convex subset of C, A be B-open, and let* $d \in \operatorname{int}(A[C])$. *Then $B \cap A^{-1}(d)$ is dense in $C \cap A^{-1}(d)$.*

If X is a locally convex space, the result can be proved using the same arguments as in the second proof of Theorem 2. Again, we can assume without loss of generality that $x = 0 \in C$, and hence also $0 \in \operatorname{int}(A[C])$. Let U be a convex open 0-neighborhood in X. Let us apply Theorem 8 to the convex set $B \cap U$, dense in $\operatorname{cl}(B) \cap U$. Since $0 \in C$ and B is dense in C, the neighborhood U contains a point of B, i.e., $B \cap U \neq \emptyset$. By the assumption that A is B-open, this implies $\operatorname{int}(A[B \cap U]) \neq \emptyset$. Using Lemma 6, we obtain $0 \in \operatorname{cor}(A[\operatorname{cl}(B) \cap U])$, and we need to prove that $0 \in \operatorname{int}(A[\operatorname{cl}(B) \cap U])$. But this follows from (2), since $A[\operatorname{cl}(B) \cap U] \supset A[B \cap U]$ is solid.

In the last step we used the relation

$$\operatorname{cor}(\operatorname{cl}(A[B \cap U])) \subset \operatorname{int}(A[B \cap U]), \tag{3}$$

which follows from (2), see Remark 12.

To cope with the general situation where X is not locally convex, we need an appropriate extension of relation (3). This extension is established in the next proposition, which could be of independent interest.

Proposition 11. *Let $M \subset X$ be convex, and let $V \subset X$ be balanced and such that* $\operatorname{int}(A[M \cap V]) \neq \emptyset$. *Then*

$$\operatorname{cor}(\operatorname{cl}(A[M \cap V])) \subset \operatorname{int}(A[M \cap (V + V)]). \tag{4}$$

Proof. Modeling on the proof of (2) given in [5, p. 59], we first prove that for every $t \in [0, 1)$

$$t\operatorname{cl}(A[M \cap V]) + (1 - t)\operatorname{int}(A[M \cap V]) \subset \operatorname{int}(A[M \cap (V + V)]). \tag{5}$$

Let $p \in \operatorname{int}(A[M \cap V])$. Then $(1 - t)(\operatorname{int}(A[M \cap V]) - p)$ is an open neighborhood of $0 \in Y$ and hence

$$\begin{aligned} t\operatorname{cl}(A[M \cap V]) &\subset tA[M \cap V] + (1 - t)(\operatorname{int}(A[M \cap V]) - p) \\ &\subset tA[M \cap V] + (1 - t)A[M \cap V] - (1 - t)p \\ &\subset A[M \cap (V + V)] - (1 - t)p. \end{aligned}$$

Here we used the fact that since M is convex and since V is balanced,

$$tA[M \cap V] + (1 - t)A[M \cap V] \subset A[M \cap (V + V)].$$

To see this, observe that if $x, y \in M \cap V$ then $tx + (1 - t)y \in M \cap (V + V)$. Thus we have shown that if $p \in \operatorname{int}(A[M \cap V])$ then

$$t\operatorname{cl}(A[M \cap V]) + (1 - t)p \subset A[M \cap (V + V)].$$

In view of the fact that the left-hand side of (5) is clearly open, this proves (5).

To finish the proof, let $x \in \operatorname{cor}(\operatorname{cl}(A[M \cap V]))$ and let $p \in \operatorname{int}(A[M \cap V])$ be distinct from x. Then we can find $y \in (x, 2x - p)$ such that $[x, y] \subset \operatorname{cl}(A[M \cap V])$. It is not difficult to see that then $x = ty + (1 - t)p$, for some $t \in (0, 1)$. By (5), this implies $x \in \operatorname{int}(A[M \cap (V + V)])$, which completes the proof. □

Remark 12. Proposition 11 generalizes (2). In particular, setting $X = Y = V$, letting A be the identity map, and assuming $\operatorname{int}(M) \neq \emptyset$, relation (4) reduces to

$$\operatorname{cor}(\operatorname{cl}(M)) \subset \operatorname{int}(M). \tag{6}$$

This relation can be viewed as the crux of the identities in (2) in the sense that they are simple consequences of (6). This follows immediately from the inclusions

$$\operatorname{int}(M) \subset \operatorname{cor}(M) \subset \operatorname{cor}(\operatorname{cl}(M)) \subset \operatorname{int}(M),$$

$$\operatorname{int}(M) \subset \operatorname{int}(\operatorname{cl}(M)) \subset \operatorname{cor}(\operatorname{cl}(M)) \subset \operatorname{int}(M),$$

$$\operatorname{int}(\operatorname{cl}(M)) \subset \operatorname{cor}(\operatorname{cl}(M)) \subset \operatorname{int}(M) \subset \operatorname{int}(\operatorname{cl}(M)).$$

Proof of Theorem 10. Suppose $x \in C$. As in Section 2, we assume without loss of generality that $x=0$. We must prove that $B \cap A^{-1}(0) \cap U \neq \emptyset$, i.e., $0 \in A[B \cap U]$, for every open neighborhood U of $0 \in X$. Let V be a balanced open neighborhood of $0 \in X$ such that $V + V \subset U$. Since $0 \in C \subset \mathrm{cl}(B)$, obviously $B \cap V \neq \emptyset$, hence $\mathrm{int}(A[B \cap V]) \neq \emptyset$ by the assumption that A is B-open. Furthermore, $0 \in \mathrm{int}(A[C])$ gives $0 \in \mathrm{int}(A[\mathrm{cl}(B)]) \subset \mathrm{cor}(A[\mathrm{cl}(B)])$, hence also $0 \in \mathrm{cor}(A[\mathrm{cl}(B) \cap V])$ by Lemma 6.

Applying Proposition 11 to the convex set $M=B$, we obtain $\mathrm{cor}(\mathrm{cl}(A[B \cap V])) \subset \mathrm{int}(A[B \cap (V+V)])$. Finally, $A[\mathrm{cl}(B) \cap V] \subset \mathrm{cl}(A[B \cap V])$ implies $0 \in \mathrm{int}(A[B \cap (V + V)]) \subset A[B \cap U]$. □

Remark 13. Theorems 8 and 10 remain valid if the interiors of the considered sets are replaced with relative interiors. As explained in Remark 7, to prove these slight generalizations, we can use the above theorems, where in the infinite-dimensional case we would identify X with the space $\mathrm{cl}(\mathrm{span}(C))$ and Y with $\mathrm{cl}(A[\mathrm{aff}(B)]) = \mathrm{cl}(\mathrm{span}(A[C]))$.

Remark 14. Recall that a map A is called open (onto Y), if it maps open sets in X onto open sets in Y. If A is open relative to a *convex* set $M \subset X$, then A is open. To see this, first note that if A is M-open, then $\mathrm{int}(A[M]) \neq \emptyset$. Hence, we can assume without loss of generality that $0 \in M$ and $0 \in \mathrm{int}(A[M])$. As seen in the proof of Theorem 10, this implies $0 \in \mathrm{int}(A[M \cap U]) \subset \mathrm{int}(A[U])$ for every open neighborhood U of $0 \in X$. Consequently, A is open.

In particular, A is X-open if and only if it is open, which explains our terminology. Moreover, if A is M-open, and $x \in M$ is such that $Ax \in \mathrm{int}(A[M])$, then $Ax \in \mathrm{int}(A[M \cap (x+U)])$ for every open neighborhood U of $0 \in X$. If Y is finite-dimensional, then A is M-open if and only if $\mathrm{int}(A[M]) \neq \emptyset$.

4. Sufficient conditions for openness relative to a convex set

As we have already mentioned, the assumption that A is open relative to B is stronger than the openness of A onto Y. Hence, it is natural to ask if openness of A is sufficient for A to be B-open under certain conditions on B. This is indeed the case if B is solid, as shown in the following proposition.

Proposition 15. *Let $B \subset X$ be convex,* $\mathrm{int}(B) \neq \emptyset$, *and let A be an open map onto Y. Then A is B-open.*

Proof. Let $U \subset X$ be open in X such that $B \cap U \neq \emptyset$. The convex set B is solid in X, hence $B \subset \mathrm{cl}(B) = \mathrm{cl}(\mathrm{int}(B))$. Therefore, the open set $\mathrm{int}(B) \cap U$ must also be nonempty. Since A is an open mapping, this implies $\emptyset \neq \mathrm{int}(A[\mathrm{int}(B) \cap U]) \subset \mathrm{int}(A[B \cap U])$. □

Remark 16. A data point $d \in A[\mathrm{int}(B)]$ is usually called a *Slater point*, see [1,7]. Under the assumptions of Proposition 15 it is clear that $A[\mathrm{int}(B)] = \mathrm{int}(A[B])$.

The notion of a *strong interior data point* was introduced in [1]. According to the definition there, a point $d \in Y$ is called a strong interior data point if there exists an $x \in B$, $Ax = d$, such that $d \in \mathrm{int}(A[B \cap (x + U)])$, for every open neighborhood U of $0 \in X$. It follows that the interior and the strong interior data points of a convex set B are the same, provided that A is B-open, see Remark 7. In particular, if Y is finite-dimensional, each interior data point is a strong interior point, as was also shown in [1].

Unfortunately, in many applications the assumption $\text{int}(B) \neq \emptyset$ is not plausible, see the discussion in [7] and the examples in the subsequent sections. However, as suggested by the examples, B could be solid in a linear subspace S of X endowed with a topology that is stronger than the topology induced by X. Thus, the space S equipped with this topology τ, say, must be continuously imbedded in X. As the next theorem shows, this idea of an "intermediate" space S gives rise to a sufficient condition for A to be B-open, in the case where $A|_S$ is an open map with respect to τ.

Theorem 17. *Let S be a topological vector space, with topology τ, continuously imbedded in X. Suppose that the restriction $A|_S$ of A on S is an open map onto Y, and that the convex set $B \subset S$ is such that its interior $\text{int}_\tau(B)$ with respect to τ is nonempty. Then A is B-open.*

Proof. We denote the closure of $M \subset S$ with respect to τ by $\text{cl}_\tau(M)$. Let us first mention that $\text{cl}(\text{cl}_\tau(M)) = \text{cl}(M)$, since τ is stronger than the topology on S induced by X.

Since $\text{int}_\tau(B) \neq \emptyset$, the convex set B is solid in S with respect to τ, hence $\text{cl}_\tau(B) = \text{cl}_\tau(\text{int}_\tau(B))$. Taking closures in X on both sides of this relation results in $\text{cl}(B) = \text{cl}(\text{int}_\tau(B))$. Now let $U \subset X$ be open in X such that $B \cap U \neq \emptyset$. There exists $b \in \text{int}_\tau(B) \cap U = \text{int}_\tau(B) \cap (U \cap S)$, hence also an τ-open set V in S such that $V \subset B \cap U$. Since $A|_S$ is an open mapping with respect to τ, this implies $\emptyset \neq \text{int}(A[V]) \subset \text{int}(A[B \cap U])$. $\square$

Combining Theorems 10 and 17, we obtain the following result.

Theorem 18. *Let the assumptions of Theorem 17 be satisfied, let B be dense in C, and let $d \in \text{int}(A[C])$. Then $B \cap A^{-1}(d)$ is dense in $C \cap A^{-1}(d)$.*

To apply this result in a concrete situation, the openness of the map $A|_S$ with respect to τ has to be ensured, i.e., an open mapping theorem is needed. We refer to [14] for a collection of such results.

It is known that the open mapping theorem holds true for F-spaces, that is, for complete, metrizable topological vector spaces [14, p. 77]. The resulting statement, as used in our examples, is formulated in the following theorem. For ease of reference, we explicitly collect all the needed assumptions.

Theorem 19. *Let the following assumptions be satisfied:*

X is a topological vector space, Y is an F-space,
A is a continuous linear map from X to Y,
$C \subset X$ is nonempty, B is convex and dense in C,
S is an F-space, with topology τ, continuously imbedded in X,
$A|_S$ maps S onto Y,
$B \subset S$ is such that $\text{int}_\tau(B) \neq \emptyset$.

Then $B\cap A^{-1}(d)$ is dense in $C\cap A^{-1}(d)$ whenever d is an interior data point, i.e., $d\in \mathrm{int}(A[C])$.

Proof. Note first that since τ is stronger than the induced topology on S by X, $A|_S$ is continuous. Moreover, $A|_S$ maps the F-space S onto the F-space Y. Hence, $A|_S$ is an open map by the open mapping theorem. The convex set B is solid in S with respect to τ, hence the map A is B-open by Theorem 17. The assertion of the theorem now follows from Theorem 18. □

Remark 20. In the finite-dimensional case, the Singer–Yamabe Theorem, i.e., Theorem 2 corresponding to $C=X$, is also of interest in the special case where B is a dense subspace of X. The Singer–Yamabe Theorem asserts in this case that an approximation (denseness) result is automatically also a result on simultaneous approximation and interpolation. We refer the reader to [4], for a number of interesting applications. It will be interesting to discuss Theorem 19 in the case where $C=X$ and where B is a dense subspace of X. First, it follows from the assumptions in the theorem that the intermediate space S is identical to B. Moreover, to be able to employ the Open Mapping Theorem, we must also require that $A|_S$ maps onto Y. In other words, we need an approximation property (S dense in X) and also interpolation ($A|_S$ maps onto Y) to obtain a result about interpolation and simultaneous approximation. Remark 26 gives an example illustrating this point. Since we think that the special case $C=X$ is interesting in its own right, we state the corresponding result as a corollary, which can be viewed as an infinite-dimensional generalization of the Singer–Yamabe Theorem.

Corollary 21. *Let X be a topological vector space, let S be an F-space continuously imbedded and dense in X, and let Y be an F-space. Let A be a continuous linear map on X that maps S onto Y. Then for every $d\in Y$, $S\cap A^{-1}(d)$ is dense in $A^{-1}(d)$. Equivalently, for every $x\in X$ there exists an $s\in S$ such that $As=Ax$ and, moreover, the set of all such elements s approximates x, i.e., x is an accumulation point of that set.*

5. Existence of smooth monotone extensions to boundary data

We first establish some notation. We shall assume that Ω is a domain of $\mathbb{R}^s$, $s\in\mathbb{N}$, i.e., an open subset of $\mathbb{R}^s$. $C^k(\bar{\Omega})$, $k\in\mathbb{N}$, will denote the space of all functions continuously differentiable in Ω up to order k and such that all the partial derivatives can be continuously extended to $\bar{\Omega}:=\mathrm{cl}(\Omega)$. Similarly, $C^{k,\dots,k}(\bar{\Omega})$ is the space of all functions continuously differentiable up to coordinate-wise order k.

In the following, we give an alternative proof of an extension theorem due to Dahmen, DeVore and Micchelli [3]. For simplicity we restrict ourselves, as in [3], to the bivariate case and we let Ω be the unit square $(0,1)^2$. We point out however, that the case when Ω is a certain bounded convex domain in $\mathbb{R}^s$, $s\geqslant 2$, e.g., the unit cube, can be treated by our method in a similar way. We will next consider nondecreasing functions on $\bar{\Omega}$. A continuous function $f\in C(\bar{\Omega})$ is *nondecreasing* (*increasing*) if $f(x)\leqslant f(y)$ ($f(x)<f(y)$), whenever $y-x\in\mathbb{R}^s_+\setminus\{0\}$, $x,y\in\bar{\Omega}$. The cone of all nondecreasing continuous functions on $\bar{\Omega}$ is denoted by $\mathrm{mon}(C(\bar{\Omega}))$. The set $\mathrm{mon}(C(\partial\Omega))$ is defined similarly.

Theorem 22. *Let* $f_i \in C^k([0,1])$, $1 \leqslant k \leqslant \infty$, $i=1,\dots,4$, *be strongly increasing functions, i.e.*, $f_i'(t) > 0$, $t \in [0,1]$, $i = 1,\dots,4$, *such that*

$$f_1(0) = f_2(0), \quad f_2(1) = f_3(0), \quad f_3(1) = f_4(1), \quad f_4(0) = f_1(1)$$

and

$$f_3(t) > f_1(t), \quad f_4(t) > f_2(t), \quad t \in [0,1]. \tag{7}$$

Then there exists a function $f \in \operatorname{mon}(C^{k,k}(\bar{\Omega}))$ *such that for all* $t \in [0,1]$

$$f(t,0) = f_1(t), \quad f(0,t) = f_2(t), \quad f(t,1) = f_3(t), \quad f(1,t) = f_4(t). \tag{8}$$

Proof. Let us first assume $k < \infty$. In order to be able to use Theorem 19, it will be convenient to define

$X = \{ f \in C(\bar{\Omega}) : \ f|_{\partial\Omega} \in C^k(\partial\Omega) \}$,
$C = \operatorname{mon}(C(\bar{\Omega})) \cap X$,
$S = C^{k,k}(\bar{\Omega})$, $B = C \cap S$
$Y = C^k(\partial\Omega)$,
$A : X \to Y$ defined as $Af = f|_{\partial\Omega}$, $f \in X$.

The notation $C^k(\partial\Omega)$ designates the space of functions f continuous on $\partial\Omega$, and such that f is a C^k function on each smooth part of $\partial\Omega$, i.e., on each of the four sides of the unit square. We endow X with the norm defined by

$$\|f\|_X := \|f\|_{C(\bar{\Omega})} + \sum_{i=0}^{1} \|f(\cdot,i)\|_{C^k[0,1]} + \sum_{i=0}^{1} \|f(i,\cdot)\|_{C^k[0,1]}, \quad f \in X.$$

By $\|\cdot\|_{C(\bar{\Omega})}$ and $\|\cdot\|_{C^k[0,1]}$ we have denoted the usual norms in the respective spaces $C(\bar{\Omega})$ and $C^k[0,1]$. Moreover, the topology on Y will be defined by the norm

$$\|g\|_Y := \sum_{i=1}^{4} \|g_i\|_{C^k[0,1]}, \quad g \in Y,$$

where g_i, $i = 1,\dots,4$, are the four pieces of the function $g \in Y$ defined in accordance with the convention (8). The norm on S is the usual $C^{k,k}$-norm.

In the proof we shall apply Theorem 19. Therefore, we have to verify that all the needed assumptions are satisfied.

The spaces X, S, Y are Banach spaces and the space S is clearly continuously imbedded in X. Furthermore, A and $A|_S$ are continuous. The density of $B = C \cap S$ in C follows from the fact that (bivariate tensor-product) Bernstein polynomials of a function defined on a rectangular domain preserve monotonicity and converge uniformly to the function along with its derivatives on the boundary. Using blending, i.e., Boolean sum interpolation, each function $g \in Y$ can be interpolated, in the sense of (8), by a function in S. Hence, $A|_S$ maps onto Y. Obviously, B has nonempty interior in S, since $k \geqslant 1$.

Finally, we need to know what it means that $d \in \operatorname{int}(A[C])$. By [3, Theorem 3.1.], if $\Omega \subset \mathbb{R}^s$ is a bounded convex domain, each nondecreasing continuous function on $\partial\Omega$ can be extended to a nondecreasing *continuous* function on $\bar{\Omega}$. In our terminology, this result can be stated as $A[C] = \mathrm{mon}(C(\partial\Omega))$. Consequently, the function $f|_{\partial\Omega}$, as given by the functions f_i, $i = 1,\dots,4$, belongs to $\mathrm{int}(A[C])$, since the functions f_i, $i = 1,\dots,4$, are strongly increasing and (7) is assumed.

Hence, by Theorem 19 there also exists a function $f \in B = C \cap S$, which is a nondecreasing $C^{k,k}$ function interpolating the boundary data f_i, $i = 1,\dots,4$. The above proof works also if $k = \infty$, in which case the corresponding topologies cannot be defined by a norm. They are defined by a family of seminorms in the usual manner. Note that Theorem 19 can still be applied since the underlying spaces are F-spaces. $\square$

Remark 23. Taking B as the cone of strongly increasing functions in S, one can also prove the existence of a strongly increasing smooth extension of $f|_{\partial\Omega}$.

Remark 24. In [3], the existence of an increasing *analytic* extension of analytic boundary data is also established. It is not clear whether this result is covered by our approach. On the other hand, unlike the result in [3], Theorem 19 implies that the set of all $C^{k,k}$ interpolants to the boundary data f_i, $i = 1,\dots,4$, can in fact also approximate any nondecreasing continuous extension arbitrarily well.

Remark 25. The problem of extending monotonically data given on the *edges* of a cube in $\mathbb{R}^3$, rather than on the boundary, is considered in [6]. The existence of smooth increasing extensions for smooth, strongly increasing edge data can be established by our methods as well, using Theorem 11.

Remark 26. The obtained result seems to be interesting even if the monotonicity requirement is dropped. In this case, each function $f \in X$, i.e., f continuous on $\bar{\Omega}$ and smooth on the boundary $\partial\Omega$, can be approximated on $\bar{\Omega}$ by smooth functions and simultaneously interpolated on the boundary.

6. Existence of smooth shape preserving extensions of functions on bounded domains

Let Ω be a bounded domain in $\mathbb{R}^s$ and let $f \in C^k(\bar{\Omega})$, $1 \leqslant k \leqslant \infty$. It is known that the function f can be extended to a C^k function over $\mathbb{R}^s$ [15]. In our terminology that means $A|_S$ is onto (see later). Using our results, we can prove more, namely that there actually exists an extension that preserves some additional shape properties of the function f such as nonnegativity, monotonicity, or convexity. In the case of nonnegativity it is not difficult to show that there exists a continuous extension that preserves the nonnegativity of f. Namely, the Tietze's extension theorem [13] guarantees the existence of a continuous extension $\hat{f}$ of f to $\mathbb{R}^s$. But then $\hat{f}_+(x) := \max\{0, \hat{f}(x)\}$, $x \in \mathbb{R}^s$, represents a nonnegative continuous extension of f. For the monotonicity case this follows directly from the Nachbin's extension theorem in partially ordered spaces [8]. In the case of a convexity preserving continuous extension it requires some extra effort to prove this, but since our purpose is mainly illustrative we will not dwell upon this here.

Next, we consider only the existence of smooth monotonicity preserving extensions. Notice that the case of nonnegativity is simpler since the cone of nonnegative functions has nonempty interior, unlike the cones of monotone or convex functions.

Theorem 27. *Let Ω be a bounded domain of $\mathbb{R}^s$, $s \in \mathbb{N}$, and let $f \in C^k(\bar{\Omega})$, $1 \leqslant k \leqslant \infty$, be strongly increasing, i.e., the directional derivative $D_t f(x) > 0$ for every $x \in \Omega$ and every $t \in \mathbb{R}^s_+ \setminus \{0\}$. Then there exists a function $\hat{f} \in \mathrm{mon}(C^k(\mathbb{R}^s))$ such that $\hat{f}|_\Omega = f$.*

Proof. As in the previous example, it will be convenient to reformulate the problem in accordance with the general setting of Section 4. We set

$$\begin{aligned}
&X = \{g \in C(\mathbb{R}^s):\ g|_{\bar{\Omega}} \in C^k(\bar{\Omega})\},\\
&C = \mathrm{mon}(C(\mathbb{R}^s)) \cap X,\\
&S = C^k(\mathbb{R}^s),\ B = C \cap S,\\
&Y = C^k(\bar{\Omega}),\\
&A : X \to Y \text{ defined as } Af = f|_{\bar{\Omega}},\ f \in X.
\end{aligned}$$

We consider the topology on X to be defined by the following family of seminorms:

$$\|f\|_{K,r} := \|f\|_{C(K)} + \|f\|_{C^r(\bar{\Omega})}, \quad f \in X,\ 1 \leqslant r \leqslant k \quad (< k \text{ if } k = \infty),$$

where K is a compact set in $\mathbb{R}^s$. The topologies on S and Y are the usual C^k-topologies. Hence, the spaces X, Y, S are F-spaces.

The denseness of $B = C \cap S$ in C can be shown in a similar way as in [7, Remark 8]. The remaining assumptions in Theorem 19 follow from the introductory remarks made above, and their verification is left as an exercise. □

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